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A first passage time distribution for a discrete version of the Ornstein–Uhlenbeck process

Hernán Larralde

Centro de Ciencias Físicas, UNAM, Apdo Postal 48-3, CP 62251, Cuernavaca, Morelos, Mexico

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Abstract

The probability of the first entrance to the negative semi-axis for a one-dimensional discrete Ornstein–Uhlenbeck (O-U) process is studied in this work. The discrete O-U process is a simple generalization of the random walk and many of its statistics may be calculated using essentially the same formalism. In particular, the case in which Sparre-Andersen's theorem applies for normal random walks is considered, and it is shown that the universal features of the first passage probability do not extend to the discrete O-U process. Finally, an explicit expression for the generating function of the probability of first entrance to the negative real axis at step n is calculated and analysed for a particular choice of the step distribution.

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1. Introduction

First passage processes [1–5] are central in many of the applications of random walks in physics (for example in Kramer's problem [1]), chemistry (in reaction kinetics [2–4]), economics (in ruin problems [5]), biology (in neuron threshold models [4]) and many other areas of research. This work concerns a first passage property of a simple generalization of the random walk, which was recently introduced as a discrete time counterpart of the Ornstein–Uhlenbeck (O-U) process [6]. Specifically, the problem of the first passage to the negative real axis at the nth step, given that the initial value is at the origin, is addressed. For normal random walks this probability distribution was shown by Sparre-Anderson to be 'super universal', in the sense that it is independent of the step distribution as long as it is symmetric and continuous [5]. However, by direct calculation of the first entrance to the negative axis at the second step, it is shown that the universality of Sparre-Andersen's theorem does not extend to the discrete O-U process. Thus, its detailed form will depend explicitly on the choice of the step distribution. Unfortunately, the general solution for the discrete O-U process requires the solution of a 'deformed' Wiener–Hopf equation which seems hard to find. However, a closed-form solution for the generating function of this probability is obtained for a specific choice of step distribution and some of its properties are analysed.

2. The discrete O-U process

The discrete O-U process we are concerned with in this work is presented in [6]. However, for the sake of completeness, we briefly describe the basic properties of this process. The process x_n is defined as follows,

$$x_{n+1} = j_{n+1} + \gamma x_n \qquad x_{n=0} = x_o \tag{1}$$

where the steps j_n are iid variables with zero mean and common probability distribution denoted by $\phi(j)$; γ is a constant [6]. The probability distribution function of x_n satisfies the following recursion relation:

$$P_{n+1}(x) = \int_{-\infty}^{\infty} \phi(x - \gamma y) P_n(y)\, \mathrm{d}y. \tag{2}$$

This process may be thought of as an unbiased random walk that takes place on a potential, which acts by changing the landing position y of the nth step to γy, the initial position of the $(n+1)$th step. In this interpretation, the walker jumps to a certain position, then 'slides' down the potential before jumping again, and so on. If the motion while sliding is assumed to be overdamped, then γy (with $\gamma \geqslant 0$) is the result of sliding down an appropriate harmonic potential for a fixed time interval τ. Thus, in this interpretation the distribution $P_n(x)$ is actually the probability density of the landing sites of this walk. However, in this work we will be concerned with first passages from the origin to the (strictly) negative semi-axis, for which the distinction between landing sites, the walker's position and its departure sites is irrelevant (indeed, the sliding motion mentioned above cannot take the walker across the origin). The continuous O-U process can be easily recovered as a continuous limit of this discrete O-U process [6].

It is clear that the process will behave very differently, depending on whether $|\gamma|$ is greater or smaller than one: if $|\gamma| > 1|$ the process diverges exponentially, while if $|\gamma| < 1$ the process is, in some sense, confined and it approaches a stationary state. Of course, when γ is exactly equal to one, the usual random walk is recovered. In what follows we will only consider $0 \leqslant \gamma \leqslant 1$.

Equation (2) can be dealt with by a Fourier transform, as is the case for normal random walks, which casts it into the recursion relation

$$\hat{P}_{n+1}(\theta) = \hat{\phi}(\theta)\hat{P}_n(\gamma\theta) \tag{3}$$

with initial condition $\hat{P}_0(\theta) = \mathrm{e}^{\mathrm{i}\theta x_o}$. The solution to this recursion relation is seen to be

$$\hat{P}_n(\theta) = \mathrm{e}^{\mathrm{i}\gamma^n\theta x_o} \prod_{m=0}^{n-1} \hat{\phi}(\gamma^m\theta) \qquad n = 1, 2, 3, \ldots \tag{4}$$

from which transport statistics of this process can be calculated.

3. First passage probabilities

We now turn to the problem of evaluating the first passage probabilities for this process. In particular, attention is focused on the probability of first entrance to the negative real axis at the nth step, starting from the origin. The interest in this problem stems from the fact that the corresponding result for the normal random walk, obtained by taking $\gamma = 1$ in the discrete O-U process, is independent of the step distribution as long as this distribution is symmetric and continuous. This is the content of the *celebrated* (according to Feller) Sparre-Anderson theorem; yet it is relatively unknown in the physics community [3, 5]. On the other hand, the trivial limit $\gamma = 0$ also gives rise to a universal first entrance probability under the same conditions. The question then arises as to whether this universality holds for all values of the parameter γ. Sparre-Anderson's original proof of the theorem was based on a relatively complicated combinatoric argument [5] which cannot be applied to the discrete O-U process. However, the first passage probability can also be obtained by purely analytical methods which can be extended to the discrete O-U process. Unfortunately, the resulting Wiener–Hopf equation appears to be intractable except in the limits $\gamma = 0, 1$. Nevertheless, for a very particular choice of the jump distribution, it is possible to untangle the equations and obtain a closed form for the generating function of the first passage probability and analyse its behaviour.

Following Feller, now denote by $P_n(x)$ the probability density of finding the particle at position x at step n when no entrance to the negative axis has occurred. Similarly, denote $Q_n(x)$ as the probability density for the position of the first entrance at step n. For the discrete O-U process under consideration these distributions satisfy the following equations,

$$P_{n+1}(x) = \begin{cases} \int_0^\infty \phi(x-\gamma y)P_n(y)\,\mathrm{d}y & \text{if} \quad x \geqslant 0 \\ 0 & \text{if} \quad x < 0 \end{cases} \tag{5}$$

and

$$Q_{n+1}(x) = \begin{cases} \int_0^\infty \phi(x-\gamma y)P_n(y)\,\mathrm{d}y & \text{if} \quad 0 > x \\ 0 & \text{if} \quad x \geqslant 0 \end{cases} \tag{6}$$

subject to the initial conditions $P_0(x) = \delta(x)$ and $Q_0(x) = 0$. The crucial point of the above equations is that the distributions are different from zero only on non-overlapping supports, and it should be emphasized that the origin is included in the support of $P_n(x)$. Clearly, the integrals in the equations above can be extended to $-\infty$, and both equations may be added, which results in

$$P_{n+1}(x) + Q_{n+1}(x) = \int_{-\infty}^{\infty} \phi(x-\gamma y)P_n(y)\,\mathrm{d}y. \tag{7}$$

Defining the generating functions in the usual way, namely $P(x,z) \equiv \sum_{z=0}^{\infty} z^n P_n(x)$, etc we can write

$$P(x,z) + Q(x,z) - \delta(x) = z\int_{-\infty}^{\infty} \phi(x-\gamma y)P(y,z)\,\mathrm{d}y. \tag{8}$$

And finally, Fourier transforming this expression leads to

$$\hat{P}^{+}(\theta,z) + \hat{Q}^{-}(\theta,z) = 1 + z\hat{\phi}(\theta)\hat{P}^{+}(\gamma\theta,z). \tag{9}$$

The + and − superscripts are reminders that the functions correspond to Fourier transforms of distributions with support in $[0,\infty)$ and $(-\infty,0)$ respectively. The solution of the above Wiener–Hopf equation leads to the generating function of the probability of first entrance at step n, $\tau(n)$ through

$$\tau(z) = \int_{-\infty}^{0} Q(x,z)\,\mathrm{d}x = \hat{Q}(\theta=0,z). \tag{10}$$

Sparre-Anderson's result is recovered if $\gamma = 1$, in which case, we can rewrite equation (9) as

$$\ln\hat{P}^{+}(\theta,z) + \ln(1 - z\hat{\phi}(\theta)) = \ln(1 - \hat{Q}^{-}(\theta,z)). \tag{11}$$

By expanding the logarithm, we see that

$$\begin{aligned}\ln(1 - z\hat{\phi}(\theta, z)) &= -\sum_{n=1}^{\infty} \frac{z^n}{n}\hat{\phi}^n(\theta) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\theta x}\phi^{*n}(x)\,\mathrm{d}x \\ &= -\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}\theta x}\phi^{*n}(x)\,\mathrm{d}x - \sum_{n=1}^{\infty} \frac{z^n}{n} \int_{0}^{\infty} \mathrm{e}^{\mathrm{i}\theta x}\phi^{*n}(x)\,\mathrm{d}x \\ &= L^-(\theta, z) + L^+(\theta, z). \end{aligned} \tag{12}$$

This procedure explicitly decomposes the logarithm as the sum of the Fourier transforms of functions defined on the domains $(-\infty, 0)$ and $[0, \infty)$, corresponding to the + and − functions in equation (9). But $\phi^{*n}(x)$ (the superscript denotes n convolutions) is the probability density for the position of an unrestricted random walk starting at the origin, with step distribution $\phi(x)$, after n steps. Thus, if $\phi(x)$ is symmetric, at $\theta = 0$ we have

$$\int_{-\infty}^{0} \phi^{*n}(x)\,\mathrm{d}x = 1/2 \tag{13}$$

and

$$\ln(1 - \tau(z)) = \ln(1 - \hat{Q}(\theta = 0, z)) = L^-(\theta = 0, z) = \tfrac{1}{2}\ln(1 - z) \tag{14}$$

or, equivalently,

$$\tau(z) = 1 - \sqrt{1 - z} \tag{15}$$

which is Sparre-Anderson's result [5]. On the other hand, the trivial limit $\gamma = 0$ does not require this elaborate machinery; however, for thoroughness, the distribution is derived from the present formalism. In this case, the Wiener–Hopf equation (9) can be separated as

$$\hat{P}^+(\theta, z) = 1 + z\hat{\phi}^+(\theta)\hat{P}^+(0, z) \qquad \hat{Q}^-(\theta, z) = z\hat{\phi}^-(\theta)\hat{P}^+(0, z) \tag{16}$$

where

$$\hat{\phi}^+(\theta) = \int_0^{\infty} \mathrm{e}^{\mathrm{i}\theta x}\phi(x)\,\mathrm{d}x \qquad \hat{\phi}^-(\theta) = \int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}\theta x}\phi(x)\,\mathrm{d}x. \tag{17}$$

Evaluating equation (16) at $\theta = 0$ we obtain $\hat{P}^+(0, z) = 2/(2 - z)$ and

$$\tau(z) = \hat{Q}^-(\theta = 0, z) = \frac{z}{2 - z} \tag{18}$$

as was to be expected.

The calculation of the first passage probability for arbitrary $0 \leqslant \gamma \leqslant 1$ requires the general solution to the Wiener–Hopf equation (9), a feat that could not be accomplished. On the other hand, as far as the question of universality goes, it is easy to see that it does not hold for arbitrary values of γ and that $\tau(n)$ depends on the specific choice of $\phi(x)$. Indeed, since $\phi(x)$ is symmetric, the probability of first entrance at the second step is given by

$$\tau(2) = \int_0^{\infty} \mathrm{d}x_1 \int_0^{\gamma x_1} \mathrm{d}x_2 \phi(x_1)\phi(x_2) \tag{19}$$

which can be readily evaluated for various choices of $\phi(x)$. Thus, if $\phi(x)$ is the uniform distribution on $[-a, a]$, then

$$\tau_{\mathrm{uniform}}(2) = \frac{1}{4} - \frac{\gamma}{8} \tag{20}$$

while if $\phi(x)$ is Gaussian we obtain

$$\tau_{\mathrm{Gaussian}}(2) = \frac{1}{4} - \frac{1}{2\pi}\arctan(\gamma). \tag{21}$$

Clearly both expressions reach the correct universal values 1/4 and 1/8 at $\gamma = 0$ and $\gamma = 1$ respectively, but they are different for all other values of γ. This is sufficient to prove that the universality of Sparre-Andersen's result does not extend to arbitrary values of γ.

4. A solvable case

As mentioned earlier, it is possible to obtain a complete solution for a particular choice of the step distribution. Namely $\phi(j) = \frac{1}{2\lambda}\,\mathrm{e}^{-\lambda|x|}$. In what follows, the generating function for the first passage probability for this case is derived, as it illustrates well enough the kind of beasts that are involved in this problem, and some of its consequences are analysed.

First of all, it is clear that the first passage probability is invariant under changes of length scale, thus, without loss of generality we take $\lambda = 1$. The starting point is equation (8), which can be written as

$$\begin{aligned} P(x,z) &= \frac{z}{2}\,\mathrm{e}^{-x} + \frac{z}{2}\int_0^\infty \mathrm{e}^{-|x-\gamma y|}P(y,z)\,\mathrm{d}y \qquad (x \geqslant 0) \\ Q(x,z) &= \frac{z}{2}\,\mathrm{e}^{x} + \frac{z}{2}\,\mathrm{e}^{x}\int_0^\infty \mathrm{e}^{-\gamma y}P(y,z)\,\mathrm{d}y \qquad (x < 0) \end{aligned} \tag{22}$$

where, for convenience, the initial condition of the recurrence relation was explicitly taken to be $\frac{1}{2}\,\mathrm{e}^{-|x|}$ at $n = 1$, which is consistent with $P_{n=o}(x) = \delta(x)$. Now, noting that

$$\frac{1}{2}\int_0^\infty \mathrm{e}^{-|x-\gamma y|-\alpha y}\,\mathrm{d}y = \frac{1}{2}\frac{\mathrm{e}^{-x}}{\alpha-\gamma} - \frac{\gamma}{\alpha^2-\gamma^2}\,\mathrm{e}^{-\frac{\alpha x}{\gamma}} \tag{23}$$

we can write

$$P(x,z) = \sum_{m=0}^{\infty} A_m(z)\,\mathrm{e}^{-\frac{x}{\gamma^m}} \tag{24}$$

and substitute in equation (22) to obtain

$$\sum_{m=0}^{\infty} A_m(z)\,\mathrm{e}^{-\frac{x}{\gamma}} = \frac{z}{2}\,\mathrm{e}^{-x} + \frac{z}{2}\sum_{m=0}^{\infty} A_m(z)\left[\frac{\mathrm{e}^{-x}}{\frac{1}{\gamma^m}-\gamma} - \frac{2\gamma}{\frac{1}{\gamma^{2m}}-\gamma^2}\,\mathrm{e}^{-\frac{x}{\gamma^{m+1}}}\right]. \tag{25}$$

Equating the coefficients of the exponentials in the equation above gives rise to a recursion relation for the coefficients $A_m(z)$:

$$A_m(z) = -\frac{z}{\gamma}\frac{\gamma^{2m}}{1-\gamma^{2m}}A_{m-1}(z) \tag{26}$$

with

$$A_0(z) = \frac{z}{2} + \frac{z}{2}\sum_{m=0}^{\infty}\frac{A_m(z)\gamma^m}{1-\gamma^{m+1}}. \tag{27}$$

The recursion relation for the coefficients $A(z)$ can be solved in terms of $A_0(z)$ and substituted into expression (27), which results in a simple equation for $A_0(z)$. This procedure yields the explicit formulae

$$A_0(z) = \frac{z}{2 - z\left[\frac{1}{1-\gamma} + \sum_{m=1}^{\infty}(-z)^m \frac{\gamma^{m^2+m}}{(1-\gamma^{m+1})\prod_{\nu=1}^{m}(1-\gamma^{2\nu})}\right]} \tag{28}$$

and

$$A_m(z) = (-z^m)\frac{\gamma^{m^2}}{\prod_{\nu=1}^{m}(1-\gamma^{2\nu})}A_0(z). \tag{29}$$

These results furnish an explicit expression for $P(x,z)$ with which we can calculate $Q(x,z)$, integrate it, and obtain the generating function for the first passage probability:

$$\tau(z) = \frac{z}{2}\frac{1 - z\left[\frac{\gamma}{1-\gamma^2} + \sum_{m=1}^{\infty}(-z)^m\frac{\gamma^{(m+1)^2}}{\prod_{\nu=1}^{(m+1)}(1-\gamma^{2\nu})}\right]}{1 - \frac{z}{2}\left[\frac{1}{1-\gamma} + \sum_{m=1}^{\infty}(-z)^m\frac{\gamma^{m^2+m}}{(1-\gamma^{m+1})\prod_{\nu=1}^{m}(1-\gamma^{2\nu})}\right]}. \tag{30}$$

Amazingly, these ungodly sums are familiar in the field of q-series [7, 8], and it is relatively easy to show that the above expression can be neatly rewritten as

$$\tau(z) = \frac{z(\gamma z, \gamma^2)_\infty}{(\gamma z, \gamma^2)_\infty + (z, \gamma^2)_\infty} \tag{31}$$

where the symbol $(z, q)_\infty$ represents $\prod_{\nu=0}^{\infty}(1 - zq^\nu)$. It is instructive to check that the above expression recovers the correct universal limits at $\gamma \to 0, 1$. Clearly, as $\gamma \to 0$, $(\gamma z, \gamma^2)_\infty \to 1$ and $(z, \gamma^2)_\infty \to 1 - z$, which indeed yields the correct expression. To verify the limit $\gamma \to 1$, we invoke the following theorem [7, 8],

$$\lim_{q\to 1^-} \frac{(q^\lambda x, q)_\infty}{(q^\mu x, q)_\infty} = (1 - x)^{\mu-\lambda} \tag{32}$$

uniformly on compact subsets of $\{x \in \mathbf{C} : |x| \leqslant 1, x \neq 1\}$. Thus

$$\lim_{\gamma\to 1^-} \tau(z) = \frac{z}{1 + \sqrt{1 + z}} = 1 - \sqrt{1 - z} \tag{33}$$

which is, again, Sparre-Anderson's result.

The generating function for the first passage time, equation (31), can be used to evaluate the statistics of this quantity. Thus, for example, the mean first passage time for this process $\langle n\rangle_\gamma$ is given by

$$\langle n\rangle_\gamma = \lim_{z\to 1} \frac{\mathrm{d}\tau(z)}{\mathrm{d}z} = 1 + \frac{(\gamma^2; \gamma^2)_\infty}{(\gamma; \gamma^2)_\infty}. \tag{34}$$

Higher moments can be obtained as well using the formula

$$\langle n^m\rangle_\gamma = \lim_{z\to 1} \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \tau(z). \tag{35}$$

Unfortunately, there appears to be no closed expression for the coefficients of the power series expansion of $\tau(z)$, thus no explicit formula for $\tau(n)$ can be provided (and it seems that neither Euler nor Gauss was very interested in generating asymptotic methods for the coefficients of q-series).

The small n behaviour of $\tau(n)$ can be calculated directly by use of the 'q-binomial theorem' [8], by virtue of which we can obtain

$$\frac{(z, \gamma^2)_\infty}{(\gamma z, \gamma^2)_\infty} = \sum_{k=0}^{\infty} \frac{(\gamma^{-1}, \gamma^2)_k}{(\gamma^2, \gamma^2)_k} (\gamma z)^k \tag{36}$$

where $(x, q)_k = (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{k-1})$ [8]; and then inverting the resulting series. This procedure yields

$$\begin{aligned} &\tau(1) = 1/2 \qquad && \tau(2) = \frac{1}{4(1+\gamma)} \\ &\tau(3) = \frac{1}{8(1+\gamma^2)} \qquad && \tau(4) = \frac{\gamma^2 + 3\gamma + 1}{16(1+\gamma)^2(1+\gamma^3)}, \text{ etc.} \end{aligned} \tag{37}$$

The asymptotic analysis for the large n behaviour of $\tau(n)$ will be undertaken below; at this point, we anticipate that this analysis requires the determination of the singularities of $\tau(z)$. Inspection of equation (31) shows that the singularities are simple poles at the zeros of the denominator $D(z) \equiv (\gamma z, \gamma^2)_\infty + (z, \gamma^2)_\infty$. Thus, since $0 < \gamma < 1$

$$D(z) = \prod_{\nu=0}^{\infty}(1 - z\gamma^{2\nu+1}) + \prod_{\nu=0}^{\infty}(1 - z\gamma^{2\nu}) > 0 \qquad \text{for} \quad z < 1. \tag{38}$$

However, it is also clear that $D(\gamma^{-1}) < 0$ and therefore the first zero occurs at $1 \leqslant \zeta_1 \leqslant \gamma^{-1}$ (the next zero occurs at $\gamma^{-2} \leqslant \zeta_2 \leqslant \gamma^{-3}$ and so on). Thus, the large n behaviour of $\tau(n)$ is an exponential decay for all $\gamma < 1$.

For any further advance, useful approximations to the infinite products are needed. We begin by considering the expression $f(z,\gamma) \equiv (z,\gamma^2)_\infty/(\gamma z,\gamma^2)$, and consider the simpler case $\gamma \ll 1$ first. In this case, all factors containing orders γ^2 or greater may be neglected in equation (36), and we are left with

$$f(z,\gamma) \approx 1 - (1-\gamma)z - \gamma z^2. \tag{39}$$

It should be remarked that this expression still gives place to a generating function of a well-normalized probability function that can be inverted explicitly. This yields an approximate expression for the first passage probability distribution:

$$\tau(n) \approx \begin{cases} 1/2 & \text{for} \quad n = 1 \\ (1-3\gamma)\left(\frac{1+\gamma}{2}\right)^n & \text{for} \quad n > 1 \end{cases} \tag{40}$$

where terms of order γ^2 have been neglected.

The situation is slightly less simple if γ is close to 1. We begin by noting that

$$f(z,\gamma)f(\gamma z,\gamma) = 1 - z. \tag{41}$$

Since $f(z,\gamma)$ is analytic at $z = 0$ and $f(0,\gamma) = 1$, we can expand

$$\ln f(z,\gamma) = \sum_{m=1}^{\infty} a_m z^m \tag{42}$$

and, using identity (41), find

$$a_m = -\frac{1}{m(1+\gamma^m)}. \tag{43}$$

This expression allows us to sort out the singularities of $f(z,\gamma)$. As we are now principally interested in an approximation for the behaviour of this function when $\gamma \to 1^-$, we can express

$$\ln f(z,\gamma) = \ln\frac{1-z}{\sqrt{1-\gamma z}} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{z^n\gamma^n(1-\gamma^n)}{n(1+\gamma^n)} \tag{44}$$

where the sum is well behaved for $z < \gamma^{-1}$ and vanishes as $\gamma \to 1$ at fixed z. Finally, writing $z = \mathrm{e}^{-s}$ and $\gamma = \mathrm{e}^{-\epsilon}$ the sum can be approximated by the integral [9]

$$\sum_{n=1}^{\infty}\frac{z^n\gamma^n(1-\gamma^n)}{n(1+\gamma^n)} \sim \int_0^\infty \mathrm{e}^{-2(1+s/\epsilon)x}\tanh(x)\frac{\mathrm{d}x}{x} = \ln\left[\frac{s+\epsilon}{2\epsilon}\right] + 2\ln\left[\frac{\Gamma\left(\frac{s+\epsilon}{2\epsilon}\right)}{\Gamma\left(\frac{s+2\epsilon}{2\epsilon}\right)}\right]. \tag{45}$$

Thus, when γ is close to one, we obtain

$$f(z,\gamma) \sim \frac{1-z}{\sqrt{1-\gamma z}}\left(\frac{\ln z}{2\ln\gamma} + \frac{1}{2}\right)^{1/2}\frac{\Gamma\left(\frac{\ln z}{2\ln\gamma} + \frac{1}{2}\right)}{\Gamma\left(\frac{\ln z}{2\ln\gamma} + 1\right)} \equiv \frac{1-z}{\sqrt{1-\gamma z}}G(z,\gamma). \tag{46}$$

Presumably less clumsy approximations to the sum can obtained (for example, by considering more terms of the Euler–McLaurin formula for the sum), but for the purposes of this work the above expression will suffice. In particular, it should be noted that this expression has the correct limit when $\gamma \to 1$ at fixed z, since, in this limit

$$\frac{\Gamma\left(\frac{\ln z}{2\ln\gamma} + \frac{1}{2}\right)}{\Gamma\left(\frac{\ln z}{2\ln\gamma} + 1\right)} \sim \left(\frac{\ln z}{2\ln\gamma}\right)^{-1/2} \tag{47}$$

thus $G(z, \gamma) \to 1$. Finally, for fixed $\gamma < 1$, the approximation of (46) can be extended to z within $0 \leqslant z < 1/\gamma$.

Using expression (46) we can find an estimate for the mean first passage time as given in equation (34), for which we obtain

$$\langle n\rangle_\gamma = 1 + \frac{1}{f(\gamma,\gamma)} \sim 1 + \frac{\sqrt{\pi}}{2}\left(\frac{1+\gamma}{1-\gamma}\right)^{1/2}. \tag{48}$$

While this result was obtained under the assumption that γ was close to 1, it is remarkably accurate even for small values of γ. In the limit $\gamma = 0$ the above expression gives 1.886, which is only 6% off the exact value $\langle n\rangle_0 = 2$.

Now we return to the exponential behaviour of $\tau(n)$ at large n when γ is close to 1. To this end we estimate the value of the first zero of $D(z)$. Using expression (46), writing $\zeta_1 = \gamma^{s_1-1}$ and $\gamma = \mathrm{e}^{-\epsilon} \approx 1 - \epsilon$ we require the solution to

$$1 + \frac{\epsilon(s_1-1)}{\sqrt{\epsilon s_1}}\left(\frac{2}{\pi s_1}\right)^{1/2} \approx 0 \tag{49}$$

where we used the fact that $\Gamma(x) \sim 1/x$ as $x \to 0$ and $\Gamma(1/2) = \sqrt{\pi}$. Thus the first zero of $D(z)$ occurs at $\zeta_1 \sim \gamma^{-1+\sqrt{2(1-\gamma)/\pi}} \sim 1 + \epsilon - \sqrt{\pi/2}\epsilon^{3/2}$; and $\tau(n)$ decays as ζ_1^{-n} for large enough n.

5. Summary and concluding remarks

In this work, we have presented a description of the calculation of the probability of first entrance to the negative semi-axis for the discrete O-U process. By studying this problem with the same conditions which apply to Sparre-Andersen's theorem, we have found that this first passage probability does not share the universality properties of that result. Thus, an explicit calculation of this first passage probability is required for each choice of jump distributions. The general solution to this problem does not seem to be simple. However, a closed-form expression for the generating function of the first passage probability can be obtained if the jump distribution is the double exponential. This calculation and some of its consequences, which unsurprisingly predict an exponential decay of the first passage probability, were presented in some detail. Other first passage probabilities for this choice of jump distribution can be carried out in essentially the same way, yet they give rise to even messier expressions than those found in this work; though some of them might be of interest, for example, the first passage probability from the origin to the region beyond a point x, which would be a counterpart of the Kramers problem for this process. At this point it is worth remarking that the reason this particular distribution is amenable for solution is the probability distribution for the sites of first entrance is a renewal distribution, and hence, it is known exactly (this also suggests that the results obtained in this work for the double exponential jump distribution could have been found using martingales [10]). Having noted this, it becomes apparent that other interesting statistical properties for the discrete O-U process with this particular step distribution may also be amenable to solution. One such property is the extension of the arcsine law [5], which is currently in progress. However, a more satisfying achievement would be to find the general solution to the Wiener–Hopf equation for arbitrary jump distributions.

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